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THE PSYCHOLOGY OF THE SIMPLE ARITHMETICAL PROCESSES: A STUDY OF CERTAIN HABITS OF ATTENTION AND ASSOCIATION.

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INTRODUCTION.

It is my purpose in the following paper to present the results of an experimental study of the simple processes of addition, subtraction, multiplication and division. These processes have been studied as they take place in the minds of that large class of people whose dealings with numbers in the ordinary affairs of life do not involve a high degree of practice. The study is therefore an inquiry into the psychology of the processes of adding, subtracting, multiplying and dividing as practically unchanged products of primary education.

In the experiments that follow eight fellow students in the Department of Psychology at Clark University (Drs. Fred Kuhlmann and Edmund B. Huey; Messrs. F. A. Lombard, L. M. Terman, J. R. Jewell, A. A. Cleveland, W. F. Book, T. Kuma) together with the writer, have served as subjects.[1] All are college graduates, but without special mathematical training. Kn. is strongly visual and auditory in mental type, H. is strongly motor but visualizes fairly, L. is slightly more auditory than visual but also has relatively strong visual and motor imagery; he is strongly impressed with the exactness of the numerical processes. T. and B. are predominantly motor with auditory imagery prominent, visual less so. J. is strongly motor; auditory imagery very prominent, visual exceedingly weak; has had experience in book-keeping. C. is predominantly visual; motor and auditory imagery also prominent; has had experience as a primary school teacher and some as a book-keeper. Bk. is strongly motor; auditory and visual imagery less pronounced; recalls "looking up answers in the back of the book." In later experience this lack of independence has developed the habit of proving results. K. is strongly motor and auditory, but visualizes fairly; earlier number training was received in the primary schools of Japan.

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[1] The writer wishes to express his obligation to these gentlemen, to the University and its Faculty in general, and to Professor E. C. Sanford in particular, under whose direct supervision the study was made.

Great care was taken to avoid both fatigue and effort, conscious or unconscious, to increase the efficiency by the practice involved in the experiments. The time was always taken chronographically to 0.1 sec., though only a subsidiary use of the results will be made in this paper.

I. ADDITION.

There were two series of experiments in adding: one in which the subjects proceeded by single digits; and the other in which combinations of digits were possible.

SINGLE-DIGIT ADDING.

A pack of about twenty cards, a single digit written upon each, with a cover card, was held by the subject in his left hand. At a signal he removed the cover card with his right hand and saw the first figure, removed the card carrying that, saw the second figure, added it; and so on to the end of the series, handling the cards as a card player would do in running through the pack to see that all were there.[1] The removal of the cards was practically automatic from the first and much more rapid than the adding. The figure to be added always appeared with the readiness to add and without conscious effort on the part of the subject.

The figures were added silently, the total sum being announced by the subject as a signal that the series was completed. The experimenter then recorded the introspections and time. The digits 1-9 inclusive, twenty of each, evenly distributed in ten packs, differently arranged at each sitting, were added five times over in five different sittings by each subject. Kn., K., L., J., T., and B. served as subjects in this series.

Adding digits in isolated pairs differs greatly from adding digits in series. The time in the latter case is by no means as even as Arnett's results would seem to indicate.[2] A wide range of variation is usually present. The average times per digit added were as follows: Kn, 1.09 sec.; K, 1.67; L, 1.28; J, 1.00; T, 1.03; B, 1.02. Average range of variation per digit added:[3] Kn, .37; K, .67; L, .36; J, .45; T, .59; B, .39.

Imagery. The terms in which the adding was done were in all the subjects largely motor, or motor-auditory (linguistic). Visual imagery, though present, seemed less fundamental to the

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[1] The cards were highly glazed playing cards, left blank upon the face, and proved extremely easy of manipulation.

[2] Arnett: Counting and Adding. *Amer. Jour. Psy.*, XVI, 1905, 335.

[3] Each of the ten packs was added through five times. The difference between the longest and shortest time of adding through each pack was taken and the sum of these differences for the ten packs divided by 180 (number of digits).

process. Thus, Kn. whispers visibly, though not audibly, as a rule. K. has a strong movement of the lips accompanied by an almost constant strained undertone. The longer he is in reaching a result, the more intense this vocal accompaniment becomes. He constantly hears himself in his imagination pronouncing the sums. In J's adding lip movement is slight. He is conscious of holding his breath and of its expulsion at times as a prolonged "n-sound." L's adding is strongly motor accompanied by auditory imagery. Sometimes he uses the full tabular form, as "8 and 6 are 14," etc. Having to clear his throat, or cough slightly, arrests the adding.

In all cases difficulty in adding greatly increased the tendency to motor expression, which sometimes became explosive. Two of the subjects, when the results did not speedily appear, found themselves repeating the verbal formula over and over, as "54 and 9 are," until the result suddenly appeared. Any slowing of the process tends with four subjects to introduce the mental enunciation of the addition tables. In the other two, it causes a more strongly accented and more conscious saying of the results.

The partial sums were always given motor expression by all the subjects, though it did not often amount to audible speech. This was by no means a matter of mere habit or accident, but a fixed and definite part of the process, serving the useful and necessary purpose of subconsciously objectifying and holding fast the sum attained while attention was given to the next digit to be added. In the partial sum thus incipiently pronounced, the stress was on the digit rather than the ten: (to indicate accent by heavy faced type) 4**6**, 5**4**, 6**3**, etc.

The Adding Consciousness. In order to understand better what follows, it may be well at this point to forecast a general conclusion confirmed by the study as a whole. The adding process is initiated by the assumption of what may be termed an "addition set" or attitude of mind corresponding to the intention to add. The "addition set" is not, itself, in consciousness to any observable degree while the adding is in progress. Its office is to direct the association processes and hold them within the addition field.

Simple addition may be considered as a process of four stages: (1) A distinct consciousness of a number to which another is to be added; (2) the recognition of this other number; (3) the associative process leading to the sum of the two; (4) the distinct consciousness of that sum. In a continuous series (4) is obviously the same as (1) in the next step. The *recognition of the result*, initiating the innervation for its motorization;[1] and the *recognition of the digit to be*

[1] The writer uses this and other similar terms as abbreviations for "more or less complete expression by incipient movements of speech."

added, as a rule not motorized, are the focal points of attention in the adding process. The associative stage is subconscious and seems to involve the following: (1) Motorization of the results (always present). In cases of difficulty or retardation of the process this motorization tends strongly to revert to the full verbal form of the addition tables. (2) Beside the motor accompaniment, and like it subconscious, is a continuous undertone of feeling upon which is based in large measure the adder's conviction that his work is correct or incorrect.

Except in temporary uncertainty, or distraction, the attention is directed forward, the associative process being left more or less to itself. It is clear that any imperfection in either phase of the attention, or in the subconscious stage of association, may result in error. Of the three phases, the most vulnerable is the stage of association.

Special Tendencies to Error. The numbers themselves in their incidental relations to each other, order of sequence, etc., became a frequent source of errors. These occur (1) where the numbers themselves rise into clear consciousness; (2) where they are less or entirely unconscious but produce the same effects. The following cases from the protocol will illustrate the tendency in question: $63 + 7 = 70$; the thought that 7 makes 70 was confusing. $26 + 7$; because 7 and $6 = 13$, the thought of the 1 in 13 made the subject say 31. $47 + 7$: 4 got into subject's mind and he wanted to say 11. $16 + 6 = 22 + 4 = 26$: confused, because "it seemed as if there were too many 6's." $79 + 2 = 87$: $5 + 5 + 4 = 19$, etc.

The following general conditions of false association appeared:

1. Numbers and results so arranged as to suggest the multiplication series (number repeated), particularly when the results, for some time follow the multiple series, and then fall just over or under it.

2. Numbers and results suggesting the counting series, as 1's in succession; or, when the digit of the successive results and the digits to be added fall in the ascending order of the count, as $21 + 2 = 23 + 4$, etc.

3. Dissimilarity in size between the digits added and the resulting digit, as $57 + 4 = 61$.

4. Similarity between the digit of the preceding result and the digit to be added, as $82 + 2$.

5. Or, in general, when any preceding digit remains in or near the focus of consciousness (as often happens when there is uncertainty regarding the accuracy of a result, or, in case the attention has been called in a particular manner to a particular digit), such a digit is likely to displace or change the digit of the result which is, or should be, in the focus of consciousness at that instant.

6. A tendency in the adding process to run ahead of the motorization leads to a frequent form of error. The attention moves on to the digit to be added before the motorization of the previous sum has taken place, with the result that the perceived digit is substituted for the digit of the sum.

General Tendencies to Error. Numerous cases of errors due to other causes than those just mentioned were in evidence. One subject is beset by the thought, for example, that he has made a mistake two or three figures back; or there may be a sort of running comment of questions as to the why of this or that, or of criticism or conjecture as to the sum total in the series being added, etc.; or the side current of thought may have reference to things entirely remote from the matter in hand.

Aside from these cases of distraction there were also occasional instants when the mind seemed a blank. So far as the condition can be judged from introspections, the mind does not wander, but associations fail and there is complete oblivion of any sort of imagery. The phenomenon appears to be closely related, if not identical, with what Mosso terms "dispersed attention."[1] The adding simply stops and there is entire inability to proceed. It is taken up again without apparent difficulty in the same place, though the adding usually goes harder and more slowly after one of these breaks. As a frequent secondary effect the subject feels that he has lost considerable time in the pause, and is likely to make an effort to increase his speed for the remaining part of the series, which in turn may result in distraction.

The Sense of Accuracy. The ultimate criterion of accuracy appears to be affective recognition. It is distinctly a feeling and not mere intellectual assent, which may often be observed dissociated from the feeling of accuracy. In such a case it seems to the subject as if he knew the result to be right, but the feeling of accuracy, or assurance, is lacking. Throughout the adding of each series, the feeling tone of certainty, or much oftener of doubt, is a constant accompaniment.

In a previous section we have distinguished, as a necessary focal point in the adding step, the recognition of the digit to be added, following the recognition and innervation of the previous sum and at the same instant that the subconscious motorization of the latter is taking place. It is in this subconscious motor process that the feeling of assurance is lodged. It is itself normally subconscious, but in case of saying a wrong result, the feeling of error often rises into the focus of atten-

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[1]Fatigue, p. 181. Dr. Kuhlmann and I myself noted the same phenomenon in a far more marked degree in adding figures continuously at maximum rate of speed in experimenting on practice and fatigue.

tion, displacing entirely the normal recognition of the digit to be added while the error is consciously corrected. It was the general experience of the subjects that any difficulty in the adding or, even an error corrected, is almost sure to give a tone of uncertainty to the remaining part of the series. The feeling of the possibility of error often causes conscious reinstatement of parts of the series to see if the right result has been said.

General conditions augmenting the sense of accuracy are uniformity of time, or rhythm; smoothness in the series; and easy combinations. Aside from repetition of the process with stronger, or with completely audible, verbal expression, no standard of absolute certainty appeared.

The Sense of Time. In so far as the sense of accuracy or assurance in adding depends on motorization, the element of time becomes an essential condition. In general, the subjects reported that, if the adding goes too fast or too slow, the sense of accuracy decreases. In the former case the motorization of results is slurred over; the time is not sufficient. In the latter case, while there is ample time for motorization to affect consciousness giving rise to the feeling of assurance, it is counterbalanced by the fact that the attention has between its normal focal points more time to wander. Accidental arrangements of digits and results become more suggestive of false association. The mean seems most important for accuracy. Motorizing of results in exact rhythmic intervals, assisted by other motor accentuations falling at the same time, was found helpful. With all the subjects any considerable deviation in uniformity of time arising from difficult combinations was always disturbing. The subject usually feels as though he must "catch up with" or "make up" the lost time. In case of error he is often in the dilemma of feeling that he *ought not* to go ahead, but that he *must not* stop. If he stops to correct his error he is distracted and harassed by a hyperæsthesic sense of "losing time;" if he goes on he feels uncertain of his accuracy.

Relative Ease and Difficulty of Combinations. It hardly need be said that any of the digits with 1 combines with the greatest ease. This is the simplest case of an easy combination. If many 1's are scattered throughout a series, the series as a whole tends to become less conscious, and the other combinations easier to deal with.

Even numbers appeared easier to combine than odd numbers, or than odd and even, for all the subjects. An intrinsic reason for this can be seen. All even-digit combinations and results are closely interrelated by common factors (always by 2 or a multiple of 2) and this common factor 2 also relates all

such combinations and results with the direct count by 2's. In the odd-digit combinations this common factorial bond is not present. An exception is the doubling of odd digits, as $3+3$, $7+7$, etc. Here the process falls back upon the multiple series which psychologically is counting. The same may be said of $9+3$, in which the common factor 3 closely relates it to the 3-count.

The simplest case of adding is the count by 1's, where the advance is made by a fixed unit at each step. The next simplest, and exactly analogous, case is the count by 2's, to which the combining of even digits is closely related. The greater ease in dealing with all possible combinations of the four even digits (2, 4, 6, and 8), is apparently due to a subconscious resolving into, or resting back upon, the 2-count, which itself is similarly related to the count by 1's. In adding (and the same was found true of the other simple arithmetical processes) the reinforcement of association for any given step, is roughly proportional to the number of other parallel or underlying processes leading to the same result. This gives a closely knit synthetic system easily reducible into simple common elements. There is a feeling of greater familiarity, because of the greater number of easily accessible links of association. Such an association complex is lacking with the odd digits, except as noted above.

While adding odd digits to each other is, in general, harder than combining even digits, it is still harder to add even and odd digits. A series of even digits yielding odd results (the series, of course, beginning with an odd number) generally proved very disturbing to all the subjects. To one subject $8+5$ presented a standing challenge; he frequently found himself on the watch for it with a determination to "know that it was 13 if it should appear." At other times when he came upon an $8+5$ combination he felt that, having had so much trouble with it, it was nearly useless to attempt to do much with it, and so counted on the five instead of adding it. With $63+4$, after struggling a while with it, the subject counted on the four. Great difficulty was had also with $7+4$ and $9+4$ in remembering which was 11 and which 13, etc. The only exception found to the inherent difficulty of this class occurs in $6+3$ and $9+6$ related to the 3-count.

Relation of the Size of the Digit Added to the Difficulty of Combining. It is much more the size of the smaller of the two digits combined than the mere fact of large digits which determines relative ease or difficulty. The shorter the step in adding any two digits irrespective of the size of one of the digits the easier the process. It was the common method of the subject to add the smaller digit to the larger. Arnett reports the same.

Ebbinghaus's study of the memory of nonsense syllables is so applicable to some of these simple number relations that his conclusions are of interest in this connection.[1] He concludes, in substance, that in the process of impressing any series of ideas upon the mind by repetition, bonds of association are formed between all the individual members of the series. Every member of such a series acquires a tendency to bring the other members along with it when it re-enters consciousness. These bonds or tendencies are of different degrees of strength. For remote members of the series, they are weaker than for neighboring members. The associative bonds for given distances backward are weaker than for the same distance forward. The strength of all bonds increases with the number of repetitions. But the stronger bonds between neighboring members are much more quickly strengthened than are the weaker bonds between more distant members. Therefore the more the number of repetitions increases, so much stronger become these bonds absolutely and relatively to those of more separated members. On the basis of time saved in relearning six sixteen-syllable series, Ebbinghaus estimates the strength of the associative bond between contiguous and separated members of a series as follows—the percents are relative to the time required to learn the series at first:

	Time Saved.
Between contiguous members	33.3 %
Skipping one syllable	10.8 "
" two "	7.0 "
" three "	5.8 "
" seven "	3.3 "
With permutation	0.5 "

Our data shows a similar relation for the number series in the four simple processes here considered. The smaller of two digits added becomes the middle term or connecting link between the larger digit and the sum. In terms of Ebbinghaus's experiment—the strength of the associative bond between the larger of two digits and the result is inversely as the size of the smaller digit; or directly proportional to the difference between the digits; thus, in $9+2=11$, the associative bond between 9 and 11 is comparatively strong, but between 2 and 11, as $2+9=11$, it is very weak. Applying this scale, two distinct tendencies appear for which the introspections afford a considerable body of evidence: (1) The easiest combinations will be those in which the greatest disproportion between the digits exists as $7+2$, $9+2$, $8+3$. In general, combinations will be harder in the increasing order of the smaller

[1]Ueber das Gedächtnis, Leipzig, 1885. Summarized by W. H. Burnham, *Amer. Jour. Psy.*, Vol. II, pp. 587-606.

of two digits giving the same result; easier in the increasing order of the larger digit. (2) Continuing up the range of possible combinations with constantly decreasing differences to the point where the difference between the two digits combined is least, as $4 + 3$, $5 + 4$, $8 + 7$, $9 + 8$, the difficulty of combining should be greatest; and curiously enough, we seem to find here the same law of the shortest step also operative—adding by subtracting 1 or 2 in the case of 9's and 8's from the smaller of two digits, and saying the result in the 'teens, especially common in 9-combinations. With other odd and even digit combinations in this class where only a difference of 1 exists, as $8 + 7$, $6 + 5$, and much less consciously with $5 + 4$, the addition is greatly reinforced and frequently comes directly from the doubling of the larger and subtracting 1. Thus $8 + 7 = 8 + 8 - 1 = 16 - 1$, etc. It is easy to add 6 and 5 because the sum is just 1 short of the familiar doublet, $6 + 6 = 12$, and also 1 greater than the more direct $5 + 5 = 10$. While $6 + 5 = 11$ is comparatively easy, $7 + 4 = 11$, as reported by the subjects, is most difficult. To formulate a somewhat general rule:

1. The ease of combining is directly proportional to the difference between the two digits combined.

2. Where the difference between the two digits is very small, particularly where there is only a difference of 1 the combining is made comparatively easy by subtracting the difference between the larger digit and 10 from the smaller of the two digits which process becomes a cue for saying, in case it be $7 - 1$, six-"teen" instead of six.

3. The most difficult digits to combine, where no common factor is present, are those falling between the two extremes, where the difference is neither very large relatively nor very small. These are: $7 + 5$, $7 + 4$, $8 + 5$, $8 + 3$, $9 + 5$, etc. To this list of most difficult must be added "class 2" in all cases where the adding is done directly.

The 'Teens. The decimal system represents a series of 17 members, in which the strength of the bonds of association steadily decreases, as the possibilities of distance between the members increases. From 1 to 10 the associative bonds are sufficient to admit of all combinations with comparative ease and certainty. In the "teens" associations weaken. Below 10, the longest step in combining is limited to 4 members, (except $5 + 5$ which belongs to a different order, the 5-count, and is not much harder than $1 + 1$. Transcending the 1-10 range, the largest number of members which may occur between any two is 8; $9 + 9$ reinforced by the multiple series is of the nature of a count; and $10 + 10$ is also a count. $10 + 9$ or $10 +$ any digit tends rapidly to become purely a cue for the immediate saying

or writing of 19, etc. No conscious adding or combining is necessarily present. This holds for all 20's, 40's, 50's, etc., plus any digit. All results derived, in a way analogous to results in the range 1-10, as $12+4$, $13+5$, $45+4$, etc., approach in degree of ease the simple $4+2$, $5+3$, $4+5$, etc. This reduction leaves a range of results 11 to 17 inclusive, derived by combining single digits, psychologically differentiated from the lower range in respect to the character of the associative bonds. The subconscious recapitulation of the 1-10 range and of the "teens" is not only the frame-work, but very largely the substance, of all adding. The tens are transcended as a subconscious count. It is primarily with the digit relations that the adding consciousness has to do. Whether the adder is in 40's, 60's, or 90's, the consciousness of the series in this sense is very much submerged. A subject having $8+8=16+9$ had to stop and think to himself $9+6$ are 15; having $47+6$ had to stop and say almost audibly $7+6$ are 13. "All adding seems like a continuous referring to the numbers under 20," etc. Because of this constant subconscious reference, any habitual difficulty which one has in the "teens" is bound to be repeated in all adding as often as the troublesome combination recurs. Thus, if one has a strong tendency to confuse $7+4=11$ and $7+6=13$, the same difficulty is experienced with all 24's, 34's, 74's, etc., $+7$, or 37's, etc., $+4$, and similarly with all 7's$+6$ and 6's$+7$.

Summary.

1. *The adding psychosis* may be divided schematically into steps, each corresponding to a digit added. In each addition-step there are two focal points of attention and a subconscious association stage; the recognition of the digit to be added, the subconscious associative elements leading to the new sum, the recognition of this sum, which goes over into a pulse of innervation for its vocal expression (usually incipient).

The recognition of the digit to be added is not the recognition of 7 or 9, as such, but the recognition of the digit as tied to its subconscious associates. Chief of the associates just preceding and simultaneous with the focal flash is the motorizing of the previous result which at the instant of recognition is "still ringing," as the subjects often expressed it. The associates directly following comprise all that is gone through in adding the recognized digit.

2. *The subconscious fringe*, *i. e.*, the associative stage, may be wide or narrow. In proportion as it is wide, the attention may be directed to many other matters, arising either from central distraction or accidental conditions of the process. These were found to be: (a) Wandering of the attention;

irrelevant thoughts and images. (b) Slight fatigue effects where for an instant consciousness seems to be a blank and no imagery of any kind can be recalled.

3. *Accidental relations of numbers and results* may rise into consciousness as direct distractions, or produce their effect unconsciously as causes of error. (pp. 5 f.)

4. *The sense of accuracy*, or assurance, seems to be closely dependent on the subconscious motorization of the results. It is essentially a feeling and not mere intellectual assent.

5. *Any feeling of uncertainty* occurring in any part of the series usually imparts an uncertain tone to the remaining part of the series.

6. *Anticipation of results and combinations of digits* tend to arise in proportion as any particular digit or arrangement of digits is repeated.

7. *Ebbinghaus's laws of association* were found to hold in single digit adding. The reason for this appears in the fact, that adding, as Mach and others hold, is derived from the count, much evidence for which has been found in this study. The child's learning to count is analogous to Ebbinghaus's learning of nonsense syllables. Counting is purely a verbal formula, the one law being that the members always follow one another in the same order. The laws of association found by Ebbinghaus hold much more for the count than for nonsense syllables, in that, different from Ebbinghaus's standard of efficiency—being able to go through the series once or twice without error—the child goes through his series of nonsense syllables, the *one*, *two*, *three*, etc., until it is impossible to say it wrong, thus fastening upon his after experience in adding a bondage to these laws, of which the introspective data gave evidence at every turn.

8. *The decimal system* represents a series of 17 members entailing a maximum range of 8 members over which the associative bond must operate in combining the larger digits.

Combination Adding.

In this experiment the subject added 50 single columns of 20 digits each. He announced his results, which were recorded by the observer, as he proceeded. At the end of each column the observer and subject together went over the adding, noting combinations, peculiarities, errors, etc. The subjects, Kn., K., L., and J., were instructed to add as naturally as possible, making no attempt to combine beyond their pre-established habits. Kn. combined in 39% of the additions; K, 51 %; L, 54%; J, 73%. The time gained by each subject per digit over the time per digit in single-digit adding was in inverse ratio to the per cent. of combining.

During this experiment (and the evidence will grow as we proceed through the other processes) it became clear that the incipient motorizing of results as the adder proceeds is a relic of the earlier motorizing of the addition tables, standing in the adding consciousness for the fully expressed verbal form. Hence the assurance attaching to it. In the last analysis the adder is only sure that his result is right when it reduces to the tabular form as "6 and 7 are 13." A further reason why practice has unconsciously selected the more or less incipient saying of results as a permanent part of the process has already been given (p. 4). Motorizing the result objectifies and holds it subconsciously, thus freeing the focus of consciousness for the recognition of the next digit or combination to be added. In single digit adding the imagery of the recognized digit to be added was visual (directly seen). It was not motorized (except as the subject reverted to the verbalism of the tables), but combined as a visual element directly with the motorized result. In combination adding a similar division of labor between imagery is present.

Sense of the Tens. In single-digit adding the tens are transcended as a subconscious count. (p. 11). In combination adding this vague sense of progressing by 10's furnishes a stimulus, but also tends to regulate the combining habit. Thus Kn. proceeds by combinations as near 10 as possible. K. breaks the digit, or combination added, into two parts, such that one of the parts added to the digit of the preceding result will give exactly the next higher even ten. For one trained on the Japanese abacus, this procedure is as simple and direct as to separate 3 beads from 5. It is entirely visual and apparently immediate. The same is true of the recombining with 10. The Japanese method obviates the difficulties of adding in the "teens." There are no (7+4)'s, (7+6)'s or (8+5)'s, etc., to surmount. While J's conscious purpose is to combine as many digits as possible at each step, his combination must not transcend the next higher ten, a general caution with all the subjects.

Prolonged and Suppressed Motorization of the Ten. With Kn., K., and J., as the vague sense of the ten of the new result arises, its motorization begins and continues slowly until terminated by the short and forceful motorization of its digit. The duration of the prolonged motorization of the ten is determined by the time required for making the combination. But the sense of the ten alone impresses L. as incomplete; hence its prolonged motorization is inhibited. This inhibition gives rise to a feeling of strain or tension which becomes more intense the longer the new result is delayed. As the new result is motorized the sense of strain is relieved. The longer the

inhibition the more time is required to regain the feeling of equilibrium; because, as L. expresses it, "the strain is greater." The holding back of the saying of the ten of the new result often became very audible. Strained hissing sounds escape like the initial *f*, *th*, *s* and *t*-sounds of 40, 30, 60, 20, etc. Apparently it is this pent up motor expression which gives rise to the vague feeling tone of strain which L. subconsciously attributes to the mental operation as a whole. In his adding each combination is followed for a few figures by single-digit adding until the feeling of mental poise is again reinstated, when another combination is attempted.

A graphic record of the time relations in saying the results was obtained by the use of a Morse key and revolving drum. The subject's left hand rested upon the key, and he easily learned to press it automatically (with neither attention nor effort) as he announced the successive results of his adding. Kn., K., and J.[1] pressed the key so far as could be determined simultaneously, with the beginning of their saying of the long drawn out ten. The key was released not simultaneously with the saying of the digit of the sum, but a little later. This slight pause seems like a brief waiting for the sense of assurance to arise. Then followed a much longer pause, during which the sense of the new ten was arising, terminated in turn by the pressing of the key at the beginning of its slow enunciation. With L. (and in the latter part of the experiment with K. also) motorization of both digit and ten took place as a sort of explosion, accent strongly on the digit, and the drum record shows but a single stroke. This distinction of two characteristic ways of motorizing results holds for each of the four processes and for writing the results in subtraction, multiplication and division L's case would also seem to indicate (as later evidence will tend to confirm) that inhibiting motorization necessitates a longer pause for the feeling of assurance to arise.

Kinds of Errors. The more common types of errors were found to be:

1. Skipping tens. This kind of error seems to arise from a vague sense of violation of uniform progress by 10's. If the subject passes over a ten, as $18 + (6 + 6) 12 = 30$, he is liable to feel that a step has been missed and in some part of the series to drop back into a lower ten. Or, if his combination leaves him still in the same ten, the impression may arise of being a ten behind and result in substitution of a higher ten at some point in the series.

[1] A slight modification was used in J's case to accommodate the registration to his habit of using a pencil for keeping his place in the series.

2. When a digit is combined or added out of its natural order, the skipped digit is liable to be forgotten and not added; or, because of the ease with which the skipped digit is later combined with an even ten, it may be added almost unconsciously, but the feeling of its not having been added may persist so strongly that it is added a second time.

3. Momentary holding over in motorization of the ten and suppression of the digit of the result, while seeking to combine with it some other digit just ahead, is liable to result in the error of merely motorizing the digit instead of combining it with the suppressed digit.

4. A relatively small class of errors arises from the subconscious attempt to correct some slight error which has supposedly been made by adding or subtracting 1 from some digit (8 added as 7 or 9). The adder is liable to add when he should subtract, and *vice versa*. Errors of this kind are most common where 9's are added as "1 less" and 11's as "1 more" than 10.

Summary.

1 The main phases as summed up for single-digit adding will be found also in combination adding.

2. Two distinct types of motorizing of results were demonstrated: (a) As soon as the sense of the ten arises its motorization begins, and is prolonged while the digit part of the result is being derived either by combination or simple addition, when the long drawn out ten is terminated by the short and accentuated motorization of the digit. (b) In the second type, motorization is repressed till the idea of the complete result arises, when both ten and digit are motorized together. The repression of motorization produces in the subject a sense of strain followed by the feeling of relief as the result is motorized.

3. Four general types of errors were found.

4. After the motorization of each result a distinct pause occurs, the pause apparently affording opportunity for the sense of assurance, following upon the motorization, to arise.

II. MULTIPLICATION.

Subjects: T., H., B., and C. The numbers upon the cards used in single-digit adding were multiplied digit by digit consecutively, by each of the digits 2 to 9, inclusive, by each subject. The multiplier was announced an instant before the "ready" and "go" signals. Other conditions were the same as in single-digit adding. In the multiplying of T., H., and B. there were many traces of the tabular verbalism, and these were more prominent with the larger multipliers. With these any difficulty or halting of the association invariably threw the subject back upon the full motor expression, as, "six 8's are 48."

Any distraction of the attention did the same, as also did general fatigue. In C's multiplying hardly any trace of the verbalism was to be observed, the digit as it appeared on the card immediately suggesting the product. He found almost no difference in the ease or difficulty of the multiplying between the large and small multipliers, and his average time per digit agreed with his introspection. We should recall that C. represents a relatively high degree of practice.

Perfect certainty of individual multiplications and of the series as a whole was the rule—a radical difference from the feeling in adding. The motorizing of the results seemed, to all the subjects, essential to this feeling of accuracy. With H., B. and T. a marked tendency to run ahead of the incipient saying of the results was observed. Sometimes, apparently, the motorizing of the results was two digits behind the multiplication. The subject always felt this running ahead of the motorizing to be a danger; a uniform time per digit seemed necessary. If any result is retarded one of two dangers is liable: (1) The subject subconsciously feels that he must hurry the multiplication to avoid a break in the uniform time of motorizing the results which becomes a distraction that further hinders the rise of the desired result. In such a case the whole process was liable to be held up at that point and the subject proceed slowly for all the rest of the series. (2) If the process does not halt on the digit whose result is retarded, it frequently happens that hesitancy allows the motorizing process to overtake the multiplying process. In this case the retarded result may not be motorrized at all, the motorization passing directly to the next result. The subject in such a case feels that he must go back and say the result whose motorization has been left out, although perfectly certain, intellectually, of its correctness, the same phenomenon is to be observed in simple dividing. C., whose time was the shortest and most uniform for all the processes, did not show this tendency to run ahead of the motor series. He felt that he must say each result before passing to the next multiplication and so dispose of it.

"*The Multiplication-set.*" In adding, some indication of an "addition-set" or attitude appeared. A slight *Anregung* was also present. The "set" is much strengthened by the first steps in the series in each of the four simple processes. In simple multiplication the "set" is intimately connected with anticipation and its principal element is the multiplier. T. and B., on hearing the multiplier announced, frequently found themselves thinking over the possibilities in the upper part of the table. All of the subjects were in a state of expectancy while waiting for the multiplier to be announced and gave it special attention. C., B., T., would at times, find themselves repeating it.

During the multiplying the multiplier was an almost (if not entirely) unconscious element in the process. Even when the verbalism appeared, the multiplier seemed of little consequence. Whether the formula runs six 1's are 6; six 2's are 12, etc., or once 6 is 6, 2 times 6 is 12, etc., the multiplier in either case is psychologically the constant, and serves only as the sign of a particular set of associations.

The digit part of the result did not stand out clear and distinct in consciousness from the ten as in addition, but each product seemed to the subject a whole. The accent appeared to be solely determined by the laws of rhythm and euphony, depending on the habit of saying the tables.

Written Multiplication.

Subjects: T., H., Bk. and C. Eight examples written upon white cards, each containing all the digits differently arranged, were multiplied by each digit, 2 to 9 inclusive, by each subject. The card, held in place by the left hand, rested across a tin plate so arranged that the writing of the digits on the card pressed down the plate sufficiently to bring it into contact with a row of brass screws directly under it, thus completing an electrical circuit, and recording the time of writing each digit upon a rotating drum. The example was covered by a piece of paper removed simultaneously with the signal "go."

In general, not knowing the multiplier and not being able to anticipate made the work slower at the start than with the cards. The subject tended to start before the particular "set" was established. The general procedure of the subjects may be summed up somewhat as follows:

First a brief hesitancy while the "set" or attitude for the particular table called for is establishing itself. Along with this arises an affective tone, its character depending on the size of the multiplier. The multiplications are as easy and sure as in the card multiplying. The subject is, however, thrown back more upon the full motor expression of the tables.

While writing the digit there is often a tendency to look back upon the immediately preceding step. The writing begins with the first faint inkling of what the digit is to be, and progresses slowly until the subject is perfectly sure of his result. Often the subject will begin to write the wrong digit and unconsciously change it to the right one without lifting the pencil. This slowness is principally confined to making the first part of the digit. With the clearing up of the idea or the assurance of its correctness, the unfinished part of the digit is executed rapidly. But Bk. inhibits the writing until sure of correctness, when the digit is rapidly and forcefully executed. This method seems to favor or necessitate a slight

resting pause involving a slight sense of relief as the previously inhibited writing of the digit takes place.

With the smaller multipliers where the possibilities of carrying are reduced, the tendency is to perform the multiplying and carrying in advance of the writing. This also occurs with the larger multipliers, where the numbers to be multiplied are small. Here both multiplying, adding and carrying are so much simplified that the digits of the complete product come quickly and easily. The motorization of it is apparently reduced, and it is handed over, at once, to automatic writing, while the attention is left free to run ahead. In the first case where multipliers are large the subject seems to be "thinking with his pencil;" in the simplified case, the writing of the digit follows the actual multiplying and carrying as an entirely detached and automatic part of the process.

Two types of difficulty arose in connection with carrying: inability to carry (T.); inability to add (H., Bk.).

The cause of the difficulty in carrying is two-fold: (a) In adding, as we saw, a division of labor takes place between motor and visual imagery. A visualized or a perceived digit was combined directly with a motor foregoing result. Here there seems to be a clash of motor images. The motor ten of the foregoing product is to be added to the following motor product. The economy of the division of labor between imagery (at least as far as simple numerical relations are concerned) seems to be that a motor and a visual image of two different digits to be added, say, may exist simultaneously, one of course being much less conscious than the other at any given instant. In the case of two motor images this is impossible, for while such images may follow each other, if the attention turn back for the first, it can only be revived as the other image is displaced. This appears to be one of the chief causes of difficulty in the carrying of multiplication and will probably explain the trouble experienced with carrying generally. (b) From a feature of adding already pointed out, the tens are very subconscious, the attention being mainly upon the digit. But in multiplying the ten is quite as important as the digit. One must attend to it in order to have it for carrying. Hence arises an inconsistency in the process itself. The subject whose attention in normal adding lets the tens take care of themselves and focuses upon the digit relations, must, in multiplying, attempt to keep both ten and digit in attention while he adds, and this apparently is possible only when a third kind of motor imagery is made use of, the writing of the digit. For C., the writing of the digit served this purpose. But for the ordinary multiplier this is not the rule in so far as the other three subjects are typical of that class. If the subject adds in the nor-

mal way, letting his ten go (T.), when he needs his ten to add he will find that it has slipped away. If he remembers that he must retain the ten (H., Bk.) so as to be able to carry it on and add it to the next product, he will find it very difficult to add, as his attention is thus drawn away from the digit, which one must of necessity attend to in adding, although he can readily carry. We may expect then, in general, to find these two types among ordinary multipliers, and also a large class of characteristic errors arising from each source.

C. did not, like the other three subjects, experience particular difficulty in either carrying or adding. He recalls earlier difficulties of this nature, but practice has apparently eradicated them. In other respects, except being on the whole much more immediate, his method does not essentially differ from the foregoing account.

The degree of reversion to the more primitive tabular expression, found with all the subjects, was in general proportional to the size of the multiplier. As the size of the multiplier increases, the possibilities of carrying are also proportionally increased. In itself it is not hard to use a large multiplier, but it proves so because of the increased range in carrying. With 2 as multiplier, if any digit is carried, it is always 1; hence carrying means "count 1." With 3 also, the carried digit will often be 1 and never be more than 2; 4, as multiplier, means a range 1 to 3 in carrying; 5 is unique, in that, while its range is 1-4, the product to be added to is always 5 or 0. In case it is 0 it is a mere cue for the saying of the digit carried; no real adding is involved; hence the greater ease of 5, as compared with 4, as multiplier. The difference between the average time per digit in written multiplication and that of simple multiplying with the cards, in general, increased proportionally with the size of the multiplier.

With the 2's, 3's and 5's full motor expression does not arise in C's multiplying. As C. expresses it, the multiplication "set" is not interrupted as with larger multipliers, owing to the few possibilities of carrying. But from the interruption of the multiplication "set" which occurs with the multipliers above 5, motorizing the full tabular form is frequently necessary to reinstate the "set." All C's products are strongly motorized. The adding is a very conscious part of the process. He does not motorize the carried digit except as it is represented in the strongly motor ten—**5**7, say. The 7 is written automatically and not further attended to. Thus the attention, freed from the digit, focuses upon the ten, as the digit to be carried. Auditory imagery is especially prominent.

Summary.

1. *The imagery* is predominantly motor and auditory. Visual imagery also is present, but apparently plays a subsidiary rôle. The great predominance of motor-auditory imagery is due apparently to the difficulties of carrying (*i. e.*, motorization of results in adding, and the auditory-motor form of the multiplication tables).

2. *Carrying.* The most vulnerable part of written multiplying is the carrying. In adding we normally hold the results in terms of motor imagery while the attention passes to the following digit (visual) to be added. In carrying, the digit to be added comes first, the product to which it is to be added second; thus the conditions of normal adding are reversed.

Two types of difficulty arise from carrying: (a) As was pointed out in a previous section, the attention is most concerned with the digit relations, and very little concerned with the tens in normal adding The multiplier who adds in the usual way is in constant danger of losing the ten (digit to be added). (b) The multiplier who realizes this danger will try to retain the ten by making it more conscious, thus drafting away the attention from the digit of the result, and will experience great difficulty in adding. In type (a) adding is relatively easy, carrying difficult. In type (b) carrying is relatively easy, adding difficult.

3. *Writing the digit* is practically automatic. With practice it tends to become a detatched system of imagery, which is very submerged in consciousness. The motorization (linguistic) of the digit tends to fall away and disappear, and the writing tends to take its place. The ten more and more tends to receive all the motorization which in normal adding goes largely to the saying of the digit. Both digit and ten thus tend, with relatively high practice, to be objectified and held until each has properly functioned. The effect might perhaps be induced with a lower degree of practice, if consciously attempted.

4. *Pedagogical.* From the foregoing data, three pedagogical inferences seem warranted.

(a) The difficulties superimposed upon the relatively easy process of multiplying by carrying would be entirely obviated by the method of writing the entire products at each step (multiplier placed at the left, the writing progressing in the natural way toward the right and adding at the end of the process). Psychologically this method is much simpler than the common method involving carrying. A similar statement should also be made for the same general method applied in adding where carrying is involved, for much the same reason.

(b) The importance of thoroughly mastering the verbal formula of the tables lies in the fact that the formula may be easily revived after long lapses of practice. The tables furnish an instrument, always at hand, for the determining of any product within their range with a certainty proportional to the proficiency with which the tables have been mastered.

The multiplication tables are also a helpful standard of reference in addition, where digits are repeated, or where the adder can single out at a glance all the 8's, say, in a column. The adding may thus become much easier, and the adder absolutely sure of correctness when $8+8+8$ yield 24, because three 8's are 24, etc. The multiplication tables are also indispensable to division.

But in proportion to the number of steps in the verbalism is the number of members which must be weeded out as the process approximates immediate association. Hence the verbal formula should be of as few words and as suggestive of immediate associations as possible. Although in general such a verbalism presents more or less of a barrier to immediate association, there is some compensation in that it may operate in a very subconscious manner, thus freeing the attention for other parts of the process. In case of momentary withdrawing of the attention, fatigue, etc., this part of the process may be relied upon practically to take care of itself.

The use of tables, however, as a basis for deriving results is not only slow, but fatiguing, because, unlike visual imagery which is relatively instantaneous, motor imagery involves a distinct process.

Another element of fatigue is the inhibition of saying the tabular form aloud. The subjects frequently and quite generally reported a strained feeling in the throat accompanying the thinking in terms of the tables; the tongue pressed against the teeth, or roof of the mouth, etc. A further slight cause of fatigue is the inhibition of motorizing non-essential members in the verbalism as the tendency to immediate association begins to short circuit the process.

(c) The Multiplication Tables. Multiplication is abbreviated addition. The child should not study addition and multiplication as two distinct subjects, but understand the latter as a special case of the former, that he may make as large a use as practicable of the reinforcement which the multiple series ought, psychologically, to lend to the adding.

In so far, however, as the addition point of view enters as a subjective factor into the tables, either consciously, or subconsciously, suggested by the form six 8's etc., it is likely to introduce a new factor affecting the ease and difficulty of multiplying. It will be easier to think three 9's than nine 3's.

Thinking in terms of the tables will be harder as the multiplier becomes large with reference to the digit multiplied.

Since it is the office of the multiplier to suggest strongly, or switch the mental processes into, the particular set of associations (table) called for, it should stand first in the verbal formula. The multiplier should always be smaller than the digit multiplied. To think in terms of the smaller digit times the larger offers no necessarily greater difficulty than the immediate recognition: $9 \times 3 = 3 \times 9$. Such a method would reduce the multiplication tables about half. The larger tables would almost disappear, *i. e.*, the twelve-table would go entirely; $\frac{11}{12}$ of the eleven-table; $\frac{5}{6}$ of ten-table; $\frac{3}{4}$ of the nine-table, $\frac{2}{3}$ of the eight-table; $\frac{7}{12}$ of the seven-table; $\frac{1}{2}$ of the six-table, etc.

The cases in which both factors are the same should probably be learned as a separate table, because they form a unique series, the square and square-root system. From their tabular form they constitute a table where it is possible to reduce the verbal formula to a minimum, thus greatly favoring immediate association. In the multiplication tables 9 may mean any number from 18 to 72. In the square system 9 always means 81. We only need to say 9–81; 8–64; 7–49, etc.

The form suggested by traces of the tables which now and then rose to consciousness with our subjects and also favored by the addition point of view was, for example, six 8's–48, in which "*are*" has disappeared and *six* is tending also to disappear.

Addition and multiplication are essentially synthetic. The associative bonds operate forward. In subtraction and division the bonds of association operate backward, hence the two distinctly different classes into which the four simple processes divide.

III. SUBTRACTION.

Subjects: Kn., K., L. and J. Simple subtraction with the same packs of cards as in single-digit adding preceded written subtraction. No pack had digits adding 100, 99 being the highest sum, 79 the lowest. The subject always started at 100, subtracting each digit as it appeared from the previous remainder. Other conditions as in single-digit adding.

The subtraction "set" is harder to initiate than either the addition or multiplication "set." The interchange of motor and visual imagery is like that of adding. By Kn., L., and J., the perceived digit (on the card) is first attended to in the light of the preceding digit of the motorized remainder, which is being motorized at the instant of perceiving the digit. If the perceived digit is smaller than the digit of the previous remainder, the subtraction falling in the range 1–10, is relatively immediate. If the digit to be subtracted is larger, it leads to what

may be called "subtracting by adding." The digit seen becomes a cue to begin motorizing the next lower ten strongly. The digit to be subtracted, say 7, is perceived while the previous remainder 53, is being motorized. Following the recognition of the digit comes the recognition, $7 > 3$, which is a cue for motorizing the next lower ten, 40. Now arises a rough formula "forty-something" and 7 are $53 = 6$, hence 46. This method, very general at first, should, however, be regarded as due to a relatively low degree of practice. Three principles, which later data will further confirm, were suggested:

1. Subtraction is harder than addition. Introspective evidence for this was general and the average time per digit subtracted was longer: Kn, 2.2; K, 2.6; L, 1.9; J, 1.5 (p. 3).

Ebbinghaus's law of serial association described in case of adding (p. 9) must, from the nature of the case, operate also in subtraction. From the fact that the count is repeated forward (counting backward as a habit is hardly comparable), the associations backward are weaker than for the same distances forward, decreasing in strength as the distance between members increases.

2. For this reason, a balance will be struck between adding and subtracting. For the adding, it will naturally tip in favor of subtraction (as we found) when the distance forward is great enough to make the associative bond weaker than subtracting over a much shorter step ($8 + 7 = 16 - 1$; $9 + 8 = 8 - 1 = 7$"teen," etc.). But when associative bonds in the forward direction are stronger than in the backward direction, subtracting (or verification) by adding becomes common; so much so, that that subtraction may be regarded as a derived process from adding.

3. But the fact of subtraction as a process derived from addition has an important secondary result. The process of "subtracting by adding," while easier as far as the fundamental laws of association are concerned, is too roundabout in practice, and immediate association tends to come in more quickly. This was a marked feature of the card subtracting. A small amount of practice with all the subjects showed great gain in deriving remainders immediately and dispensing with addition. The same phenomenon also appeared in dividing, which bears a like relation to multiplication. Subtraction and division are far more immediate than addition and multiplication, though probably far less practiced. But the subject, as a rule, does not trust his remainders as they appear and only learns to do so after considerable practice. Subtraction and division, while the most immediate of the four simple processes, are also characterized by lack of confidence and consequent "proving;" and this is especially true of subtraction. As the adding gave

way to immediate association, it was still retained by Kn., L. and J. to prove the results.

K., as in adding, always breaks a larger subtrahend digit in two, in such a way that one part will give the ten; the other is then taken from 10. All his subtractions thus fall in the range 1-10. What was said in favor of his method in adding applies still more in subtracting.

Written Subtraction.

Subjects: Kn., J., L. and K. The method was similar to that used in written multiplication. Fifty examples having each digit in both subtrahend and minuend, but differently arranged each time, were performed by each subject. The more important part of the data concerns the three different methods employed. K's method is Japanese. If the subtrahend digit is larger, it is taken from 10 and the remainder added to the minuend digit. Otherwise the method is the same as that of Kn. L. and J. use the older method of borrowing by adding 1 to the subtrahend; Kn., the common method of subtracting 1 from the minuend.

No evidence of the entire suspending of motor imagery for one of the digits (usually the lower, though often both are strongly motorized, as 6 from 13) appeared; but in simple taking away, there is a tendency for a glance at the two digits, involving mere appearance without recognition of either digit in itself, to constitute a cue for immediately writing a certain figure without motorization. Several other similar cues were also found: (1) Where the two digits, subtrahend and minuend, are equal. (2) When a difference of 1 exists, the subtrahend digit being larger, this relation may become an immediate cue for writing 9. (3) The subtrahend digit being larger by 2, may become a cue for immediately writing 8. But the difference of 2 yielding 8 immediately seemed a limit. Such cues work best when there is no borrowing. In case of borrowing, it also works fairly well in the method of J. and L. But reducing the minuend digit (Kn., K.) hinders the use of such cues in proportion to the degree of forgetting whether or not borrowing should occur.

The most difficult subtracting is where the borrowing is interspersed with cases of simple taking away. The habit of borrowing rapidly establishes itself and the subject is liable to continue it over into the simple subtractions and *vice versa*. Involved in adding to (K.) or subtracting from, the upper digit (Kn.) is the necessity of remembering whether the previous minuend digit has been raised to a "teen," or looking back to see. For K. and Kn. this was most difficult. The difficulty is twofold:

1. The memory as to borrowing is very feeble when it exists at all. Sometimes it seems to be retained as a memory of moving the pencil towards the next minuend digit, from which theoretically the borrowing is to be done, and also a general feeling of having moved slightly in that direction (K.), or as a memory of the visual 1 of the digit raised to the "teen" (J.); which vaguely retained sense means: "take away 1 from the minuend digit before adding to, or subtracting from it." From difficulty of recall and the vagueness of such imagery arises a strong tendency to look back each time to the foregoing step, just before the adding or subtracting occurs. The glance back, however, interposed as it is between two closely related parts of the process, is also vague; hence, it too, is unreliable. The lower digit is already being motorized, which is really the beginning of the adding, or subtracting, and the normal direction of the attention impels one to continue: now interposed between these naturally almost inseparable processes comes a hasty backward glance. In case this glance brings with it the impression of having to borrow, the adding, or subtracting step already begun is broken off and the minuend digit reduced by 1, and then the adding, or subtracting, is completed. The only looking back which can be relied upon occurs as the digit (remainder) is being written, which takes place so automatically that the attention is free to look back.

2. The greater inconsistency, however, occurs in decreasing the minuend digit. It is not difficult to reduce this digit by 1, but a confusion is apt to arise from the fact that ordinarily this digit operates in the process of adding or subtracting as a visual element. The attention moves up to it from the lower digit and as it is recognized in connection with the lower motorized digit, the addition or subtraction occurs, with the minuend digit directly within the field of vision. Hence arises the possibility of having to visualize a 4, say, with a perceived 5 directly in the field of vision.

If the 4 is motorized, the difficulty in holding it in motor imagery with the original 5 directly within the field of vision, may lead to errors allied to that class in addition where a perceived digit displaces a motor digit in the result. The motor 4 must also displace the natural and more economic division of labor between visual and motor-auditory imagery in the following subtraction. For K., the process is further complicated by having to subtract and add to the same figure at as nearly the same time as possible. Kn. by the ordinary method must perform a double subtraction, which proved quite as difficult as the "subtract-then-add" process of K.

When both upper and lower digit are alike, the more or less conscious backward glance gives no cue as to whether borrow-

ing shall occur, which becomes a source of error, the subject assuming wrongly either alternative. J. and L. did not experience difficulties of the above nature. The diagrammatic comparison of the methods given below will make plain the reasons.

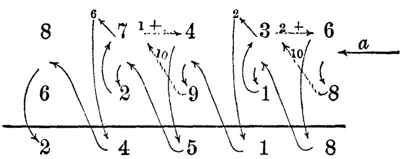

Fig. 1.

General movement of the attention. The following diagrams represent the general movement of the attention for each subject as determined introspectively. The arrows indicate the focal points in the general direction which the attention takes in moving through the subtraction steps. These are the points of conscious recognition, but, as already pointed out, there is no such phenomenon present as distinct flashes of the attention or disintegrated recognitions of the various important points in the process. Each recognition is an integral part of the one preceding; or the various subconscious effects, following the foregoing recognition, constitute the necessary associates of the following recognition. No true recognition can take place unless the accustomed preceding recognition has occurred. Like a few instances in multiplication and division, if a foregoing recognition fails to occur in its natural order, the subject finds himself, as he looks at the figures in the written form of the example, staring at symbols devoid of all meaning. The arrow (a) in K's method *Fig. 1*, indicates the first focal point, recognition of the relative size of the two digits, 6 and 8. Following this recognition the attention, indicated by the second arrow, moves to the subtrahend digit, 8, its recognition initiating strong motorization. The third and fourth arrows in broken lines indicate the vague movement of the pencil towards the 3 from which 1 is to be taken, and represent the taking of the 8 from 10, the recognition being a cue for the strongly motorized 2, which, as the attention moves to the recognition of the 6 (visual), is added to it. The fifth arrow represents the fifth focal point, the rising into focal consciousness of the resulting 8, initiating innervation both for its motorization and writing.

The next step, 3—1, may represent in principle the chief inconsistency of the method. The first arrow, indicating the recognition of the relative size of the digits, has obviated the necessity of taking 1 from 10 and adding 9 to the upper digit.

But as the attention moves to the recognition of 3, from the physiological memory of the general movement in the direction of 3 while taking 8 from 10 in the previous step, or more often looking back to see, as the 3 is recognized, the recognition involves reducing it by 1, and thinking it as 2. This requires a change in the focus of the atttention from 3 to 2. The difficulty arising, whether the 2 be visualized or motorized (p. 25), increases in proportion to the difficulty of the subtraction. The last subtraction 8—6 represents the simple case reduced to two focal points, recognition of the relation 8-6, immediately associated with the writing of 2.

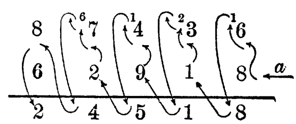

Fig. 2.

For Kn. *Fig. 2* (subtrahend motor, minuend visual), the first and second focal points, are invariable, except when the tendency to write a certain digit from the mere appearance of the two digits involved, becomes operative. The raising of 6 to 16 does not require an added pulse of attention, as does the reducing of the minuend digit. In the particular "set" of the preceding recognition 8 > 6, 6 is not recognized as 6, but as 16, a mentally visualized 1 taking its place beside the 6 as it is recognized. The difficulties of reducing the minuend digit, however, are increased in cases where the digit has to be raised to the "teen" and also reduced by 1.

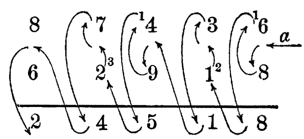

Fig. 3.

For J, *Fig. 3*, the first focal point is the recognition of the relative size of 6 and 8; the second embraces the recognition and motorization of 8; in the third focal point, 6 is recognized as at motor 16. J's subtractions are in motor terms. The fourth focal point completing the step, with 8 motorized and automatically written, the attention goes at once to the next digit, 1, which is recognized as 2 (subconscious count), the first focal point of the new step.

Slightly different is L's diagram (visual type) *Fig. 4*. His

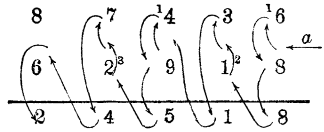

Fig. 4.

second focal point motorizes 16; the third visualizes 8, yielding the motor result 8, automatically written; the attention then goes at once to 1, raised to 2, by subconscious count. The 2 is strongly motor; but passing to 3, the visual imagery predominates. Again, in the third step, 14—9, 14 is the second focal point strongly motorized; the third focal point is 9 visualized; the fourth, the motor result 5, from which, while automatically written, the attention moves directly to the clearly recognized 3 (subconsciously raised from 2), and so on.

Certain errors and difficulties inherent in the method of borrowing commonly taught at present—the liability of forgetting whether borrowing should occur, the difficulties in reducing the minuend digit by 1, the characteristic errors and general inconvenience of looking back, together with the affective tone of doubt, which it creates at every step—are greatly reduced, if not entirely obviated, by the older method used by J. and L. When the common method is used with visual imagery, *Fig. 2*, the visualized 1 (placed before minuend digits requiring borrowing) must be entrusted to physiological memory until the next minuend digit is reached. From that digit it is separated by four focal points: the resulting digit of the remainder, the recognition of the next digit of the subtrahend, the recognition of the relative size of this digit and the corresponding digit of the minuend, and the the recognition of the minuend digit to be reduced. In case the minuend were motorized and the subtrahend digit kept in visual imagery, the attention moving downward from the upper digit to the lower, four focal points would still intervene. If the subtracting were entirely in motor terms, as "8 from 16 are," four focal points would intervene.

Let us now turn to the older method in which we find the three possible ways of proceeding represented in the diagrams of J., predominantly motor, *Fig. 3*, and L. predominantly visual, *Fig. 4*. In the first case where the upper digit is visualized as the "teens," which visualization is the basis of the physiological memory, only one focal point intervenes between such visualization and the actual increasing of the digit, as against four focal points intervening in the common method. In the case of the motorization of the minuend digit and the visualizing of the subtrahend digit, two focal points will occur. In

the other possible case, the motorizing of both subtrahend and minuend digits, only one focal point will occur. A combination of the Japanese with the older method, eliminating subtracting from the "teens" in the older method and reducing the minuend digit in the Japanese method, would seem, from the psychological point of view, a distinct advantage.

As with the schematic description of the adding consciousness and that of multiplying, so here many modifications may arise, though the main focal points of the attention will always be found present. Attention may go off in a great variety of ways between the normal focal points. It may double on its course, go back and reinstate previous focal points; it may be influenced occasionally or habitually by subconscious doubt as to whether borrowing should occur, and so go over the foregoing step again and return with the missing associates. Any case of simple subtracting may prove to be an immediate perception of relation of size, leading at once to the writing of a particular digit always associated with this particular perception. The common habit of verifying the result as it rises into clear consciousness may cause the interposing of a complete addition step; and distractions from central causes are always liable.

Writing the Digits of the Remainder. The following general principles will be found to hold, with slight modification, for multiplication and short division as well as for subtraction.

1. The writing of the digit is automatic.

2. Doubt of the result or the habit of verifying may inhibit the writing of the digit until the verifying has taken place, in which case two things may happen: (a) The digit may be quickly and forcefully written as a kind of explosion of inhibited action, giving the subject a sense of relief and inducing a slight rest, which such inhibition seems to necessitate, before going to the next step. (b) The inhibited ideo-motor action may reach expression more slowly and occur during the preliminary recognitions involved in the next step, being terminated just before the actual subtraction is made.

3. If the tendency to begin writing the digit as the first vague idea of it rises into consciousness is not inhibited, which is more common, the subject is usually repeating the process or verifying it by a reverse process, while the writing is slowly progressing.

4. If assurance comes with the first focal recognition of the digit as in simpler cases, the time of writing the digit will be the time required to perform the next step.

5. Thus, the automatic writing of the digit follows the movement of the attention. If the coming in of the focal idea with its tone of assurance is delayed, so much slower propor-

tionally becomes the writing of the digit. If it clears up in consciousness quickly, the remaining unfinished part of the digit is executed with a rapid stroke. If the train of imagery connected with the clearing up of the idea in consciousness is intercepted by irrelevant imagery, the movement of the pencil stops.

Summary.

1. Simple subtraction is a derived process, being governed in its mode of operation by the laws of association which operate in adding and which in turn are derived primarily from the conditions of counting.

2. Subtraction is harder than adding because the associative bonds operate more weakly in the reverse order. For this reason children should learn to count backward as well as forward.

3. Hence arises the phenomenon of "subtracting by adding," which in practice becomes too cumbersome, and tends to disappear, giving place to immediate association. The adding formula, however, tends to persist in subtraction as a means of verification.

4. The older method of increasing the subtrahend appears to be superior to the present (more logical) method of decreasing the minuend, which is largely responsible for the difficulties of borrowing. (p. 24.)

IV. DIVISION.

Before the experiments in written short division, a series in simple dividing took place. Eight packs of eighteen cards each were used. Each pack contained all of the nine multiples of the digits, except 1, yielding a quotient 1 to 9, each multiple appearing twice. The subject divided through each pack, each time differently arranged, ten times. The divisor was announced just before the "ready" and "go" signals Other conditions were the same as already described in the foregoing experiments. The subjects were J., Bk., H. and C.

Division is a derived process depending primarily upon multiplication in much the same way and for the same reasons as subtraction depends upon addition.

The original formulæ: as "6 into 34, 5 times, etc ," "18 divided by 3 gives 6," are only vaguely motorized as the work is performed, and apparently assist but feebly in the deriving of results, if at all In C's dividing (relatively immediate) the process still seems like a "reverse" process; and this was more pronounced in the other three subjects. Many instances of reaching the quotient digit by way of the multiplication formula occurred, as 6 "somethings" are 54 = a shadowy 9, immedi-

ately verified by transposition into the verbalism, "six 9's are 54." This double formula represents apparently the primitive method. It is always the last but sure resort when the quotient digit fails to appear promptly.

As with subtraction, this cumbersome double formula tends rapidly to be short circuited into immediate association. Even in the short practice of 1440 simple divisions, these traces, quite evident at the beginning, grew appreciably less. But unlike the addition formula in subtracting, the multiplication formula does not appear to be retained for purposes of verification. It would seem that addition, most characterized by uncertainty of the four simple processes, passes on its own uncertainty to subtraction. Although the subject almost always finds his result right after proving, he does not therefore reason, consciously or subconsciously, that the result as it appears is probably right and so rid himself of the verifying habit. The habit, on the contrary, tends to persist. The subject, in the most pronounced cases, seems to revel in the sense of the certainty, so largely lacking in its parent process, addition, which use of the adding verbalism imparts to the subtracting step. The certainty of simple multiplication, seems also to impart something of itself to simple division; and any sort of verification tends to fall away. The character of the immediate association in division is quite different, therefore, from that in subtraction.

Kinds of Errors. No errors involving digits not factors of the number divided were discovered. Any factor commonly used as a divisor of a given dividend is liable to appear as the quotient digit, or even the divisor itself may so appear. Three kinds of such errors were found:

1. Errors making the divisor or some other factor of the dividend the quotient digit, as $24 \div 8$ giving a quotient 8 or 4. Numbers such as 24, 16, 12, containing more than two factors commonly used as divisors, were especially liable to this kind of error. Such errors are insidious because the subject generally passes over them with no sense of inaccuracy. Most of them were discovered by the subject when a number immediately following gave the same result, as $24 \div 6 = 3$, followed by $18 \div 6$ also yielding 3; whereupon the subject remembers the previously motorized 3 with the vague recognition that the number which gave it was not 18. Even C. made errors of this kind.

2. Another sort of error arose in dividing a digit by itself, as $5 \div 5 = 5$.

3. When only a difference of 1 exists between the divisor and quotient digits, even if the divisor is normally present in the subconscious fringe, the subject frequently has difficulty

in selecting the required quotient. In $72 \div 8 = 9$, 8 and 9 are contiguous members in the counting series, and as one comes into consciousness it tends strongly, as earlier shown, to bring the other along with it, and this creates a sense of doubt as to which is really right. Cases of this kind are about the only instances of any tendency on the part of the subject to doubt the quotient digit as it appears.

The phenomenon of running ahead of the motorizing of the results, as in simple multiplication, appears in simple division also. But from the experience of the subjects a definite limit is evident about two recognitions in advance of the motorizing. If a third recognition occurs, the subconsciously following train of motor results is apt to be lost sight of. In such a case all becomes hazy, and the process stops. Having once gone two recognitions ahead of the motorizing series, any slowing of the recognition process is likely to result in a peculiar kind of confusion. The subject feels the subconscious motor series approaching and that he must keep ahead of it. Such a momentum has arisen from the necessity of its keeping up with the recognition series that when the recognition series slows up, this acceleration in the motor series induces forced recognitions and any of the foregoing kinds of error may result, or, as more often happens, complete confusion may arise. Conversely the checking of the motor series imposes a check upon the recognition series.

WRITTEN SHORT DIVISION.

Subjects: T., Bk., H. and C. The method and examples used were the same as described in written multiplication except that the numbers were divided instead of being multiplied.

As in the preceding cases of multiplying and simple dividing, it is essential to attend to the divisor before beginning the example. A feeling tone arises from the initial attention to the divisor corresponding to its size. Getting well started (*Anregung*) requires one or two repetitions of the division process.

In dividing, the writing of the quotient digit occurs near the beginning of the division step. In multiplication and in subtraction the recording of the corresponding digits occurs near the end of the step. After the quotient figure appears (digit to be written), the subtraction is still to take place and the new dividend to be formed. This fact gives to division a very different character from that of the other two written processes. It greatly encourages automatic writing of the digit. The writing tends to go along as a parallel but separate series, requiring no attention. As the idea of the quotient digit arises into the focus of consciousness it is motorized, but

handed over directly to the writing mechanism, while the attention, thus freed, passes to the subtraction of the product from the dividend number. The figure is written slowly, its time being determined by the time required to complete the step.

Although the double multiplication formula did not tend to persist in simple dividing, it became far more apparent in written division, especially in the more difficult cases where the dividend number is so far away from the multiple that it suggests it feebly, if at all. Let us take as a dividend 59, divisor, 8, multiple to be derived, 56, quotient-figure, 7. A vague multiplication formula appears as the subject attends to the 59 visually; as "something $\times$ 8 is something in the vicinity of 59." After more or less hesitancy 56 appears as this last wanting term. The subject may now have further difficulty in dividing 56, in any of the ways pointed out in the last section. Deriving the multiple 56 and the quotient digit 7 are both accomplished in motor terms.

If we work through a step or two of a section of an example the various phases may be more clearly illustrated. Let us take $6295 \div 8 = 786+$.

1. While the divisor 8 goes along, as a rule in the simple cases, as a very submerged part of the process, it is reinstated as often as the subject brings back the multiplication formula to determine the multiple and quotient digit. The subject has, let us say, already derived the first multiple 56, as just described, a process often characterized by the subjects as "feeling or groping about for" the number required. The 7, obtained thus by a motor procedure, is now to be written automatically. The subject begins to write the digit as soon as the first faint idea of what it is to be arises, often starting to make a wrong digit, but subconsciously changing it to the right one without removing the pencil. The 7 is motorized as the idea of it becomes focal, but it drops entirely from consciousness normally as soon as derived, and the automatic writing of it takes place as a disconnected part of the process.

2. On the instant of yielding the 7 (handed over directly to the automatic writing with the shortest and most incipient pulse of attention comporting with the degree of assurance) the strongly motor 56 is at the same instant being subtracted from the 62, directly seen and not motorized. In the subtraction, however, the minuend 6 is disregarded and 5, following the law also of all tens in adding and subtracting, is very submerged. A visual 1 has appeared at the left of the 2, the visually imaged 1 and the perceptual 2 giving 12. Thus the subtraction takes place yielding a strongly motor 6. This 6 is located below the 62. The writing of 7 is still slowly

progressing, its speed depending entirely on the time of the movement of the attention through the step.

3. The motor 6 felt as below 62, is now felt to move upward and take its place beside the 9 as a visual 6, and the motor image having been translated into visual terms, now disappears.[1] This visualized 6 and the 9 (directly seen) is the new dividend 69. The motor imagery, freed from the care of the 6, now initiates the motor process involved in getting back to 64, and from this 64 to 8, also motor, but at once handed over to the automatic writing. The same cycle is now to be repeated for the next step, and so on to the end of the example.

The motor parts of the dividing step were extremely pronounced for H., often to verifying of subtracted results by counting. Bk. represents an extreme type of the verifying habit. His quotients, often immediate, must be multiplied over again "to see if they are right," and he usually finds that they are; but he keeps on verifying as if the reverse were true. Very often remainders appear immediately, but are not to be trusted until verified by adding. He often gets his remainders directly by adding, in which case they are verified by direct subtraction. When he gets his new dividend, if it is large, it is liable to look "too large," which means that the whole step will be repeated "to see." The effect of this verifying habit upon the average time was marked.

As in multiplication, Bk. writes the figure with a quick, forceful stroke, the writing following the assurance. This induces a further disadvantageous consequence in the breaking off of the process to write the figure, for the main part of the division step is yet to be performed. It is not only bad in itself for the reason that it works against immediate association; but it is unduly fatiguing because of the excess of motor imagery which the verifying involves; it lengthens the time and prevents automatic writing. When the thread of the process interrupted by the writing of the digit, is again taken up, the product to be subtracted has often disappeared, and this involves the necessity of reinstatement. H. began by writing the quotient digit in a similar manner, but before the end of the experiment changed his method unconsciously to that already described.

Immediate Association. The type of immediate association in subtracting appears to be a little different from that where the mere appearance of two digits becomes a cue for the writing of a certain figure. Beside, not being written the digit

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[1] These subjects have never followed or been taught the method in short division of putting the remainder before the dividend figure for the new dividend.

is not an end in itself, as in actual subtraction, but a means. The point of view is that of deriving a digit for the ten of the new dividend. That is, it is immediate subtraction in the particular division "set." It partakes apparently of the kind of immediate association where attention to the multiple tends to produce in consciousness the quotient digit, the divisor being given. Let us take as an illustrative case, 6 from 11, represented by the accompanying diagram, *Fig. 5.* Although similar, the case is more complicated than attending to the multiple for deriving a desired quotient. There is, however, one compensating condition. In the case of the multiple, the given factor (the divisor) is extremely subconscious; in this case the digit is very close to the focus of consciousness, being motorized as the 11 is being attended to visually. As the 11 (the right hand digit being seen directly, the left digit visualized) is attended to, it tends to bring into consciousness some one of the digits with which it is associated as a sum. Moreover it

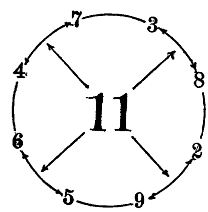

Fig. 5.

tends to bring one of the digits because a digit is wanted for the purpose of the new dividend. The 6 is already present in strong motor imagery; hence that digit will tend to appear which is the associate of 6 in producing 11; and 5 will appear all the more quickly because 5 and 6 are contiguous members in the counting series, though in descending order. Many cases were reported by all the subjects in which the digit for the new dividend seemed "to drop out" of its own accord from the given number and take its place at the left a little above the dividend digit of the new dividend. In such cases there is attention to the given visual number alone, with no special thought of subtraction, exactly as the multiple is attended to in simple division. A similar diagram (*cf.* James: Principles of Psychology, Vol. I, p. 586) might represent the type of immediate association in deriving the quotient digit. Errors in yielding other factors than the quotient digit are more liable in proportion as the presence of the divisor in the subconscious fringe is weak.

Relation of Dividend Number to Difficulty in Dividing. In multiplying, the larger multipliers seemed to increase the diffi-

culty proportionally. This was not on account of the multiplying in itself, but because of the increased possibilities of carrying and adding. In division a similar relation seems to hold, but not because of the increased possibilities of the size of the digit to be subtracted, although this fact is not without influence. The chief increase of difficulty, arises from the corresponding increase of possible distance in the number scale between the dividend and the next lower exact multiple. The reported cases and general introspective evidence of all the subjects seem to warrant the following general statement:

1. The most difficult cases of deriving a multiple occur when the dividend-number falls just under the next multiple above; as 61, divisor 7. 63 is strongly suggested, but 63 is not wanted; 56 is the desired multiple. It was the opinion of the subjects that the attempt to get back directly to the 56, was the cause of what they felt to be the worst feature of division, "the feeling, or groping, for" part of the process. The strong and natural associative bond in the direction of 63, works directly against the attempt of the subject to get back by a chance hit to 56. Persisting in the point of view established both by education and practice, the subject can only flounder about vaguely for the desired multiple. If, as sometimes happens, he takes up as a last resort, with the natural line of association, going to 63, thus setting aside for the moment the dominating point of view, the finding of the multiple becomes comparatively easy, though somewhat roundabout. At least it is little harder than the method (multiplication) in which many of the quotients are actually derived; as seven 9's are 63, but one less is wanted, hence seven 8's, etc. At least there is a pedagogic advantage in this procedure, in that it works in the line of the natural associations.

2. The easiest cases occur when the dividend-number falls just over the desired multiple. Being thus so much farther removed from the multiple above, and nearer the one below, the dividend number tends to suggest the latter.

3. A region of associative dead-lock appears to occur with the larger divisors somewhere in the middle region between the two multiples. These cases are not so difficult as the first class because associative tendencies, up and down, neutralize each other. This difficulty, of course, disappears proportionally as the divisors become smaller, because the possibilities are always 1 less than the divisor used.

In these cases of increase in difficulty with increase in the size of the divisor (and similarly in multiplying, with increase in size of the multiplier), another factor is operative. It is allied to discrimination and choice in reaction experiments. Such experiments show that as the possibilities increase the

time of reaction is lengthened. Most in point for our special problem is the study of Vintschgau[1] upon the times required in multiplication. His subjects reacted by giving the products, one of the two factors being given them in advance. As the smaller of the two factors was always announced first the subject hearing "nine" had but one possibility 9×9. If 2 was first announced, the range of possibilities varied from 2 to 9. The tables were run through in this way, and it was found that, in general, the length of time increased as the number of possibilities increased, 9×9 giving the shortest time.

Summary.

1. Division is a derived process based upon multiplication. Unlike subtraction, which still continues to be largely influenced by the point of view of addition, division tends to free itself from the point of view of multiplication and to develop a type of immediate association. It does not revert for verification to the multiplication formula, in a degree approaching the tendency in subtraction to revert to addition. While subtraction is a verifying and proving process par excellence, division is a process of immediate association.
2. The writing of the quotient figure occurs at the beginning of the division step, and this not only favors automatic writing of the digit, but practically necessitates it.
3. In written short division the step comprises three stages (pp. 33 f.).
4. The difficulty of the process as a whole increases proportionally with the size of the divisor, because of the increased range of possibilities as to the dividend numbers falling above the multiple (p. 35).

In the preceding sections I have considered chiefly the more fundamental aspects upon which there was practical unanimity in the experience of the subjects, or marked disagreement. The various experiments are of such a character as to be easily repeated. Any one wishing to put the foregoing inferences to a practical test, should have little difficulty in ascertaining to what extent these phases are present in his own experience and that of others.

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[1] *Pflüger's Archiv*, Vol. XXXVII, pp. 127-202.